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On the system of astronomical constants, by *W. de Sitter* †.

[edited and completed by *Dirk Brouwer*, with the aid of notes by *W. de Sitter*]

1. *Introduction.*

The values of the fundamental astronomical constants that are at present used in the various national ephemerides and astronomical tables are, with a few exceptions and alterations, those fixed by the international congress of 1896, and are based on NEWCOMB's work in the last two decades of the nineteenth century.

Several of the constants are capable of experimental determination in two or more different ways, independently of each other. There are, however, theoretical relations between the different constants. An ideal system of fundamental constants would be one in which these theoretical relations were satisfied *rigorously*, while the adopted value of each individual constant agreed with its observed value, within the limits of uncertainty of the latter. The accumulation of observational material and its increase in accuracy will necessitate a revision of the system from time to time, but such revisions should not be too frequent, and, above all, they should not be piecemeal, but embrace the whole system. It is of much greater importance that the value of a fundamental constant used in any individual investigation should be exactly defined than that it should be the best available at the moment.

Since NEWCOMB's work was completed several constants have been determined with greater accuracy, and of some of these, e.g. the solar parallax, the officially adopted values differ from those used in NEWCOMB's system. Further, the secular retardation of the rotation of the earth has become firmly established, and should be taken into account in the construction of the system of fundamental constants. The same remark applies to the corrections resulting from the relativity theory of gravitation in the precession of the equinoxes and the motions of the perihelia. As regards the agreement with observations the system at present in use is on the whole fairly satisfactory. The only serious discrepancy is the contradiction between the value of the mass of the moon as derived from the constants of precession and nutation and from the lunar inequality, to which attention was called by Dr. JACKSON in 1930 [1]).

On the other hand, our present system is *not consistent*. The chief inconsistency is between the adopted values of the lengths of the tropical and the sidereal year (both taken from NEWCOMB's tables of the sun), and the value of the general precession in longitude. The first corresponds to NEWCOMB's preliminary value 50″·2482 (1900·0), whilst the adopted value is 50″·2564. The value of the constant of nutation corresponding to this last value and HINKS's mass of the moon, $\mu^{-1} = 81{\cdot}53$, is $\mathbf{N} = 9''{\cdot}2139$, the adopted value being 9″·210. This latter corresponds to $\mu^{-1} = 81{\cdot}64$.

Other minor inconsistencies are the following. The adopted mass of the earth ($m^{-1} = 329390$) corresponds to the value 8″·79 of the solar parallax, while the adopted value is 8″·80. The adopted value of the constant of aberration 20″·47, on the other hand, corresponds to $\pi_\odot = 8''{\cdot}80$, at least to the last decimal place given; the exact value corresponding to $\pi_\odot = 8''{\cdot}80$, BESSEL's equatorial radius of the earth ($b = 6377{\cdot}4$) [2]) and NEWCOMB's velocity of light ($c = 299860$), is $\mathbf{k} = 20''{\cdot}475$. The value of the light-time τ corresponding to $\mathbf{k} = 20''{\cdot}47$ is $\tau = {}^{d}{\cdot}0057683$, while BAUSCHINGER's tables give $\tau = {}^{d}{\cdot}005770$ [3]), the last decimal place of which is thus inexact.

It would appear from the above that a revision of our official system of fundamental constants has become desirable. Such a revision should, however,

[1]) *M. N.* **90**, p. 742, 1930.

[2]) This is the value given in BAUSCHINGER's tables. If we substitute CLARKE's value ($b = 6378{\cdot}25$), adopted by NEWCOMB, we find $\mathbf{k} = 20''{\cdot}478$.

[3]) The second edition gives ${}^{d}{\cdot}0057706$ (BROUWER).

not be lightly undertaken. It would involve a repetition of NEWCOMB's enormous work, taking into account all the observational results and theoretical developments that have become available since NEWCOMB's time, as well as the results of investigations still in progress, such as the Eros campaign of 1930–31. In the meantime I have thought it worth while to derive a *consistent* system based on those values of a small number of mutually independent fundamental constants which appear to me to be at the present moment the most probable, expressing the other constants in terms of these fundamental ones and of hypothetical corrections to them (see the table at the end of this article). The adopted fundamental constants are eight in number, viz. the solar parallax $\pi_\odot$, the mass of the moon μ, the velocity of light c, the dynamical compression of the earth $(C - A) / C$, the mean radius of the earth R_1, the acceleration of gravity at mean latitude g_1, and the two small constants $\varkappa$ and λ_1 depending on the inner constitution of the earth [1]). In addition to these, of course, a number of other constants enter into the result, such as the mean motions and secular accelerations, the values of which are, however, so accurately known that no hypothetical corrections to them need be introduced.

The limits of uncertainty given after the sign $\pm$ are intended to be *probable* errors. In some cases these are the result of actual computation, in other cases they depend on judgment, based on the definition of the probable error, i.e. the probability that the true value is within the range between the assigned value $-$ and $+$ the probable error is considered to be one half, so that also the probability that it is outside this range is one half.

1) The statement that these constants are independent should be understood to mean that there exists no relation by which the number of fundamental constants could be reduced. The particular values that can be assigned to the constants are, of course, dependent on each other through observations. In this connection attention should especially be drawn to the dynamical flattening of the earth, $H = (C-A)/C$, which is computed from the constant of precession with the aid of the adopted value for the moon's mass (derived from the lunar inequality). As the luni-solar precession is known with a percentage probable error which is only one-fifteenth of the probable error of the mass of the moon, the error of H is almost entirely determined by the error in μ^{-1}, so that the corrections w and z to these two constants cannot be considered as independent quantities; as shown on page 225 they must fulfill the relation

$$-\cdot 6747\, z + w = 0.$$

This should be kept in mind when computing the probable errors of those derived quantities in which w and z occur with coefficients of the same order, i.e., the constant of nutation and its ratio to the constant of precession. For the same reason the probable errors attached to the values for the constants of precession are simply those estimated from direct observations, while in the computation of the error of the constant of nutation the above relation has been duly taken into account.

In all other cases, except when specially mentioned, the probable errors given are those derived from the probable errors attached to the fundamental constants.

The inconsistency mentioned could have been removed from the beginning, at the cost of some complication of many formulae, by replacing the dynamical flattening by the precession constant as one of the fundamental constants. But no choice of fundamental constants would be ideal in this respect: the number of observable functions of the constants is greater than the number of fundamental constants. However, after introducing the alterations mentioned, the results given would appear to contain no inconsistencies which are of any practical importance (BROUWER).

I. PRECESSION.

2. *Adopted masses of the planets.*

The secular variations of the elements of the earth and other planets have been computed by NEWCOMB [1]). It is easy to effect the changes in the results consequent on the introduction of other values for the masses. I adopt the following values for the reciprocals of the masses (including satellites):

	Mercury	7 500 000	$\pm$	1 500 000
	Venus	404 000	$\pm$	1 000
	Earth	327 900	$\pm$	200
(1)	Mars	3 085 000	$\pm$	5 000
	Jupiter	1 047·40	$\pm$	0·03
	Saturn	3 490	$\pm$	5
	Uranus	22 750	$\pm$	200
	Neptune	19 500	$\pm$	200

The mass of Mercury is the value used by NEWCOMB. The determination of this mass is so uncertain that it does not seem worth while to apply a correction.

The mass of Venus is the one adopted by me in 1927 as a mean between the determinations by SPENCER JONES from the sun [2]) and by ROSS from Mars [3]). A later determination by SPENCER JONES from the secular change of the obliquity [4]) agrees with it within the probable error.

The mass of the earth is that corresponding to the solar parallax $\pi_\odot = 8''\cdot 803$ (strictly speaking $8''\cdot 80328$).

The mass of Mars is an average of several determinations from the satellites.

The mass of Jupiter is the value derived by me as a mean from all determinations [5]).

1) *Astr. Pap. A.E.* V, 4, p. 377, 1895.
2) *M.N.* **86**, p. 435, 1926.
3) *Astr. Pap. A.E.* IX, 2, p. 261, 1917.
4) *Cape Annals* XIII, 3, p. 59, 1932.
5) *Cape Annals* XII, 1, p. 153, 1915.

Dr. SPENCER JONES in Cape Annals XIII, 3. p. 62, states the values of the masses of Mercury, Venus, the earth, Mars and Jupiter, "which, from all the evidence available, may be regarded as the most probable". These agree exactly with the above, with the exception of a small difference in the mass of the earth-moon system.

The mass of Saturn is an average between the different values derived by Dr. WOLTJER [1]). The probable error given is my estimate.

The masses of Uranus and Neptune are practically NEWCOMB's.

Probably the above values could be improved, and the probable errors in most cases decreased, by a careful discussion of all available observational material, but such a discussion would be very laborious, and should form a part of an exhaustive investigation of the whole problem of the elements of the planetary orbits. For our purpose the values (1) are sufficiently exact.

3. *The motion of the ecliptic.*

The motion of the ecliptic under the influence of the attraction of the other planets is defined by the quantities which NEWCOMB denotes by $\varkappa$ and L [2]). If we denote by π and Π the inclination and node of the ecliptic at any epoch on that of the standard epoch, the node being counted from the equinox of the standard epoch, we have

$$\frac{d}{dt}(\pi \sin \Pi) = \varkappa \sin L,$$

$$\frac{d}{dt}(\pi \cos \Pi) = \varkappa \cos L.$$

With the new masses (1) we find from NEWCOMB's data

		1600	1850	2100
(2)	$\varkappa \sin L$	$+\ 4''\cdot374^3$	$+\ 5''\cdot350^8$	$+\ 6''\cdot320^4$
	$\varkappa \cos L$	$-47''\cdot348^7$	$-47''\cdot073^6$	$-46''\cdot784^2$

The unit of time here is the Julian century. If T_1 is the time expressed in tropical centuries from 1850·0, we find from (2)

$\pi \sin \Pi =$	$+\ 5''\cdot350^7\ T_1$	$+\ ''\cdot19461\ T_1^2$	$-\ ''\cdot000184\ T_1^3$
p. e. by Mercury	$\pm$ 50	$\pm$ 14	$\pm$ 5
,, ,, Venus	$\pm$ 19	$\pm$ 31	$\pm$ 0
,, ,, other planets	$\pm$ 1	$\pm$ 1	$\pm$ 0
total p. e.	$\pm$ ·054	$\pm$ ·00034	$\pm$ ·000005
$\pi \cos \Pi =$	$-\ 47''\cdot072^6\ T_1$	$+\ ''\cdot05645\ T_1^2$	$+\ ''\cdot000381\ T_1^3$
p. e. by Mercury	$\pm$ 42	$\pm$ 8	$\pm$ 0
,, ,, Venus	$\pm$ 71	$\pm$ 7	$\pm$ 1
,, ,, other planets	$\pm$ 1	$\pm$ 1	$\pm$ 0
total p. e.	$\pm$ ·083	$\pm$ ·00011	$\pm$ ·000001

These formulae are the base for the development of the precessional constants in powers of the time.

4. *The general precession and the precessional constant.*

The determination of the constant of general precession is beset with many difficulties. First, of course, there is the difficulty of eliminating the solar motion and the star streams. In the case of the latter we have to make an assumption regarding the distribution of the stars over the two streams in different regions of the sky. Further, in the proper motions in right ascension it is very difficult to separate the precession from the motion of the equinox (i.e. the error of the adopted common motions in R.A. of the fundamental stars). In the case of declinations the systematic errors in the proper motions are the most serious difficulty. The proper motions in both right ascension and declination are, moreover, affected with the rotation of the galaxy. This latter is eliminated in a determination of the constant of precession from proper motions in galactic latitude, as has been done by OORT [3]) and by PLASKETT and PEARCE [4]). In this case, however, also the precession and the motion of the equinox are very badly separated, and moreover the proper motions in galactic latitude, being derived from those in α and δ, are affected with the systematic errors of these latter. It appears to me safer to keep separate the proper motions in right ascension and in declination, which each have their own, largely unknown, systematic errors. They must then, before serving for a determination of the precession, be corrected for galactic rotation, the constants of this latter being derived from radial velocities and the *periodic* terms in the proper motions.

1) *Leiden Annals* XVI, 3, p. 63, 1928.
2) *Astr. Pap. A.E.* V, 4, p. 377.
3) *B.A.N.* IV, No. 132, p. 85, 1927.
4) *M.N.* **94**, p. 704, 1934.

The uncertainty of the constant of precession due to all these complications is, however, so small that its effect on the other fundamental constants is unimportant. I therefore adopt as a convenient value of the general precession in longitude per tropical century, which is probably very near the best according to the present state of our knowledge,

$$(4)\quad p = 5026''{\cdot}00 \pm 0''{\cdot}10 \qquad (1900).$$

We put:

$\mathbf{P}$ = the precessional constant, according to NEWCOMB's definition,
$p_\circ$ = the lunisolar precession,
p = the general precession in longitude,
p_g = the geodesic precession,
λ = the planetary precession,
m = the general precession in right ascension,
n = the rate of motion of the pole of the equator.

Then we have, for the fundamental epoch,

$$(5a)\qquad \frac{d\Theta}{dt} = \frac{d}{dt}(\pi \cos \Pi),$$

$$\lambda = \operatorname{cosec} \Theta \, \frac{d}{dt}(\pi \sin \Pi),$$

and for any epoch the relations:

$$(5b)\qquad \begin{aligned} p_\circ &= \mathbf{P} \cos \Theta \\ p_1 &= p_\circ - p_g \\ p &= p_1 - \lambda \cos \Theta \\ \mathrm{m} &= p_1 \cos \Theta - \lambda \\ \mathrm{n} &= p_1 \sin \Theta. \end{aligned}$$

The geodesic precession is a direct motion of the zero of longitude along the ecliptic, of which the amount is given by the formula

$$(6)\qquad p_g = \frac{3}{2}\frac{k^2}{c^2}\frac{n}{a} = 1''{\cdot}915 \text{ [1]}.$$

For the obliquity I have provisionally adopted the correction to the secular variation determined by SPENCER JONES [2], making it $-\ 47''{\cdot}14$ per century. This gives for 1850 a correction to NEWCOMB's value of

$$\delta\Theta = +\ 0''{\cdot}15,$$

making it

$$(7)\qquad \Theta_{1850} = 23^\circ\ 27'\ 31''{\cdot}83 \pm 0''{\cdot}06.$$

The probable error attached is based on judgment.

From the values (3), (4), (6) and (7) we derive by the formulas (5)

$$(8)\qquad \begin{array}{rl} \mathbf{P} = 5493''{\cdot}158 & -\ ''{\cdot}003692\ T_1 \\ \pm\ 175 & \quad \pm\ 25 \end{array}$$

The secular variation will be derived below. Of the probable error of $\mathbf{P}$ the greater part is due to the planetary precession λ, i.e. to the uncertainty of the planetary masses.

5. *Development of the precessional quantities in powers of the time.*

From the data (3), (6), (7) and (8) we can derive the development of the values of the different quantities in powers of the time. I have used the formulas given by ANDOYER [3]. Taking the tropical century as unit of time, and denoting by T_1 the time counted in this unit from 1850·0, and by T from 1900·0, I find for 1850:

$$(9)\qquad \begin{array}{lllll}
\Theta = 23^\circ\ 27'\ 31''{\cdot}83 & -\ 47''{\cdot}073\ T_1 & -\ ''{\cdot}0087\ T_1^2 & +\ ''{\cdot}00187\ T_1^3 \\
\qquad\qquad\quad \pm {\cdot}06 & \quad \pm {\cdot}083 & \quad \pm\ 6 & \quad \pm\ 1 \\
\Theta_1 = 23^\circ\ 27'\ 31''{\cdot}83 & & +\ {\cdot}0653\ T_1^2 & -\ {\cdot}00777\ T_1^3 \\
\qquad\qquad\quad \pm {\cdot}06 & & \quad \pm\ 7 & \quad \pm\ 2 \\
p_1 = 5037''{\cdot}213 & +\ \ {\cdot}4957\ T_1 & -\ {\cdot}00004\ T_1^2 \\
\qquad \pm {\cdot}161 & \quad \pm\ 9 \\
p = 5024{\cdot}883 & +\ 2{\cdot}2336\ T_1 & +\ {\cdot}00023\ T_1^2 \\
\qquad \pm {\cdot}100 & \quad \pm\ 39 \\
\lambda = 13{\cdot}441 & -\ 1{\cdot}8958\ T_1 & -\ {\cdot}00010\ T_1^2 \\
\qquad \pm {\cdot}136 & \quad \pm\ 33 \\
\mathrm{m} = 4607{\cdot}428 & +\ 2{\cdot}8082\ T_1 & +\ {\cdot}00007\ T_1^2 \\
\qquad \pm {\cdot}094 & \quad \pm\ 48 \\
\mathrm{n} = 2005{\cdot}265 & -\ \ {\cdot}8572\ T_1 & -\ {\cdot}00037\ T_1^2 \\
\qquad \pm {\cdot}061 & \quad \pm\ 15
\end{array}$$

Θ is the angle between the instantaneous equator and the instantaneous ecliptic, Θ_1 that between the instantaneous equator and the ecliptic of the fundamental epoch.

For 1900·0 we have

[1] See Section 12, formula (49).
[2] *Cape Annals*, XIII, 3, p. 58.
[3] *Bull. Astron.* **28**, p. 67, 1911.

$$
\begin{aligned}
(9')\quad \Theta &= 23^\circ\ 27'\ 8''{\cdot}29 - 47''{\cdot}080\ T - ''{\cdot}0059\ T^2 + ''{\cdot}00186\ T^3 \\
\Theta_1 &= 23^\circ\ 27'\ 8\ {\cdot}29 \qquad + {\cdot}0607\ T^2 - {\cdot}00777\ T^3 \\
p_1 &= 5037''{\cdot}461 + {\cdot}4956\ T - {\cdot}00007\ T^2 \\
p &= 5026\ {\cdot}000 + 2{\cdot}2337\ T + {\cdot}00014\ T^2 \\
\lambda &= 12\ {\cdot}493 - 1{\cdot}8959\ T - {\cdot}00004\ T^2 \\
\mathrm{m} &= 4608\ {\cdot}832 + 2{\cdot}8081\ T - {\cdot}00004\ T^2 \\
\mathrm{n} &= 2004\ {\cdot}836 - {\cdot}8576\ T - {\cdot}00032\ T^2.
\end{aligned}
$$

The computations have been made to one additional place in order to ensure the accuracy of the last place given.

II. Mean motions and the measure of time.

6. *Secular variations.*

The variability of the rotation of the earth produces an apparent variation in the observed mean motions of the planets and the sun, according to the rule

$$\frac{n' - n}{n} = -\frac{\delta\omega}{\omega}.$$

Here n is the true mean motion expressed in an ideal invariable time unit, which we may call Newtonian time, and for which we may choose the length of the mean solar day at a fixed epoch, say 1900·0. On the other hand n' is the observed mean motion expressed in astronomical time, of which the unit is the actual mean solar day. In the case of the moon there is, in addition to this, a change of the real mean motion by the reaction of the tidal retardation of the earth's rotation on the moon. Let this be

$$\frac{n - n_\circ}{n} = M\frac{\delta\omega}{\omega}.$$

The total change in the mean motion of the moon is thus

$$\frac{n' - n_\circ}{n} = (M - 1)\frac{\delta\omega}{\omega}.$$

The ratio M must be derived from observations. These give [1]) in the longitude of the sun an empirical term $+ (1''{\cdot}65 \pm ''{\cdot}15)\ T^2$, and in the moon $+ (5''{\cdot}05 \pm ''{\cdot}30)\ T^2$. Consequently

$$\frac{\delta\omega}{\omega} = -\frac{3''{\cdot}30 \pm ''{\cdot}30}{n_\odot}\ T = -(2{\cdot}546 \pm {\cdot}230)\ .\ 10^{-8}\ T$$

$$(1 - M)\frac{\delta\omega}{\omega} = -\frac{10''{\cdot}10 \pm ''{\cdot}60}{n_{☾}}\ T = -(0{\cdot}583 \pm 0{\cdot}35)\ .\ 10^{-8}\ T,$$

from which

(10) $M = 0{\cdot}771 \pm {\cdot}084.$

The secular change of the mean motion of the earth due to the perturbations of the planets is given by Newcomb [2]). With the masses (1) it gives

(11) $\delta_1\ \mathrm{n}_\odot = -(0''{\cdot}0404 \pm ''{\cdot}0002)\ T.$

For the moon Brown's tables give a term $+ 7''{\cdot}14\ T^2 + ''{\cdot}0068\ T^3$, of which $1''{\cdot}116\ T^2$ is due to precession. I have not thought it worth while to reduce Brown's term to the masses (1). The change would have been inappreciable and the whole secular acceleration is of very small importance for our purpose. We take thus:

(12) $\delta_1\ \mathrm{n}_{☾} = +(12''{\cdot}05 \pm ''{\cdot}003)\ T + ''{\cdot}0204\ T^2.$

The probable error attached to the first term corresponds to that of the secular variation of the earth's eccentricity, which will be derived below.

We have thus the following secular terms in the centennial mean motion of the sun:

	secular perturbations	$-0''{\cdot}0404 \pm ''{\cdot}0002$
	rotation of the earth	$+3\ {\cdot}30 \pm {\cdot}30$
(13)	sidereal mean motion	$+3\ {\cdot}2596 \pm {\cdot}30$
	precession	$+2\ {\cdot}2337 \pm {\cdot}01$
	tropical mean motion	$(+5\ {\cdot}493 \pm {\cdot}30)\ T,$

and for the moon

	secular perturbations	$+12''{\cdot}05 \pm ''{\cdot}003$
(14)	rotation of the earth	$+10\ {\cdot}10 \pm {\cdot}60$
	sidereal mean motion	$(+22\ {\cdot}15 \pm {\cdot}60)\ T.$

The secular variations of the earth's eccentricity and perihelion are given by Newcomb, *Astr. Papers A.E.* V. 4, p. 377. The unit of time is the Julian century. With the masses (1) I find

	$\frac{de}{dT}$	$e\frac{d\varpi}{dT}$	e	$\frac{d\varpi}{dT}$
1600	$-8''{\cdot}429$	$+19''{\cdot}402$	$3480''{\cdot}686$	$+1149''{\cdot}76$
1850	$-8\ {\cdot}557$	$+19\ {\cdot}318$	$3459\ {\cdot}454$	$+1151\ {\cdot}81$
2100	$-8\ {\cdot}689$	$+19\ {\cdot}247$	$3437\ {\cdot}899$	$+1154\ {\cdot}77$

[1]) See *B.A.N.* IV, No. 124, pp. 20–38, 1927. We have $M = 1 - 1/Q$ in the notation of that paper. The value (10) of M thus corresponds to $Q = 4{\cdot}367$.

[2]) *Astr. Constants*, p. 187.

From these values of de/dT' and the value of e for 1850·0 from NEWCOMB's tables of the sun we have

$$(15)\quad \begin{aligned} e &= 3459''\cdot454 - 8''\cdot557\ T_1 - ''\cdot0260\ T_1^2 - ''\cdot00011\ T_1^3\ (1850\cdot0), \\ &= 3455''\cdot169 - 8\cdot583\ T - \cdot0262\ T^2 - \cdot00011\ T^3\ (1900\cdot0), \\ &\quad \pm\cdot035 \quad \pm\cdot024 \end{aligned}$$

which has been used to compute the values of e given in the fourth column above. From the fifth column we find then for the sidereal motion of the sun's perigee in a Julian century:

$$(16)\quad \begin{aligned} \frac{d\varpi}{dT'} &= +1151''\cdot81 + 1''\cdot00\ T_1 + ''\cdot073\ T_1^2\ (1850\cdot0), \\ &= +1152\cdot33 + 1\cdot07\ T + \cdot073\ T^2\ (1900\cdot0). \\ &\quad \pm 2\cdot40 \end{aligned}$$

It should be noted that in most of these perturbational terms the greater part of the probable error is due to the uncertainty of the mass of Mercury (estimated as one fifth of its amount), although the contribution of Mercury to the perturbation itself is comparatively small.

7. *Mean motions.*

The values of the mean longitude at epoch and the mean motion are also affected by the secular terms due to the rotation of the earth. Corrections to the tabular values were derived by me in 1927 [1]), and these were revised by SPENCER JONES as a part of his discussion of the occultations of stars by the moon. I adopt JONES' revised values [2]). Taking the constant and linear terms only, the correction to the longitude of the sun is

$$\Delta L_\odot = +1''\cdot42 + 3''\cdot306\ T$$
$$\pm\cdot10 \pm \cdot15$$

Adding this to NEWCOMB's tabular value, and using the quadratic terms (13), we find for the mean longitude of the sun referred to the mean equinox of date, T' being the time in *Julian* centuries [3]) counted from 1900 Jan. 0·0 (Grw. mean noon),

$$(17)\quad L_\odot = 279^\circ\ 41'\ 49''\cdot46 + 129602771''\cdot436\ T' + 2''\cdot746\ T'^2.$$
$$\pm\cdot10 \qquad \pm\cdot15 \qquad \pm\cdot15$$

By subtraction of the part due to precession, $5026''\cdot107\ T' + 1\cdot117\ T'^2$, the sidereal mean motion
$\pm\cdot10$
in a Julian century is found to be

$$129597745''\cdot33 + 3''\cdot26\ T,$$
$$\pm\cdot18 \quad \pm\cdot30$$

from which the sidereal mean daily motion is found to be:

$$(18)\quad n'_\odot = 3548''\cdot1928906 + ''\cdot0000892\ T.$$
$$\pm 50 \qquad \pm 82$$

The sidereal year is thus:

$$(19)\quad 365^d\cdot25635442 - {}^d\cdot00000918\ T,$$
$$\pm 50 \qquad \pm 84$$

and the tropical year

$$(20)\quad Y = 365^d\cdot24218946 - {}^d\cdot00001548\ T.$$
$$\pm 42 \qquad \pm 142$$

For the moon the correction to BROWN's tables is

$$\Delta L_☾ = +4''\cdot72 + 12''\cdot731\ T.$$

The sidereal mean motion in a Julian century of the moon thus becomes

$$1732559392''\cdot684 + 22''\cdot15\ T + ''\cdot0204\ T^2,$$

or the daily motion

$$(21)\quad n'_☾ = 47434''\cdot8909701 + ''\cdot000606\ T + ''\cdot00000056\ T^2.$$
$$\pm 50 \qquad \pm 16$$

8. *The measure of time.*

Astronomical time is measured by the transit over the meridian of the fictitious mean sun. NEWCOMB remarks [4]) that the mean right ascension of the fictitious mean sun differs from the mean longitude of the actual sun by a secular term $0^s\cdot020\ T^2$. This is due to the difference of the secular variations of the general precessions in longitude and in right ascension, combined with the actual secular acceleration of the sun. It must be increased by the secular acceleration due to the retardation of the rotation of the earth. NEWCOMB leaves to the astronomers of the future the question how best to meet the difficulty

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1) *B.A.N.* IV, No. 124, p. 21.

2) *Cape Annals* XIII, 3, pp. 39 and 41. It should be remembered that $S = T^2 + 1\cdot329\ T - \cdot256$.

3) In the small terms it is not necessary to make the distinction between T and T'.

4) *Astr. Const.* p. 188.

thus arising. Evidently the solution is to alter the definition of the fictitious mean sun, and, instead of defining it with NEWCOMB as "a point on the celestial sphere having a uniform sidereal motion in the plane of the earth's equator and a Right Ascension *as nearly as may be* equal to the sun's mean longitude", to define it as a point of which the right ascension is *exactly* equal to the sun's mean longitude. Its sidereal motion will then not be strictly uniform. It is found by subtracting from the mean motion corresponding to (17) the general precession in right ascension in a Julian century, viz:

$$4608''\cdot93 + 2''\cdot808\ T.$$

The sidereal motion of the fictitious mean sun in a Julian century is thus found to be

$$(22)\qquad n''_{\odot} = 129598162''\cdot51 + 2''\cdot684\ T$$
$$\pm\ 25 \quad \pm\ 300$$

The right ascension of the fictitious mean sun affected by aberration now becomes

$$(23)\ L = 18^h 38^m 45^s\cdot 933 + 8640184^s\cdot847624\ t' + {}^s\cdot1831\ T^2,$$

where t' is the time counted in Julian years from 1900 Jan. 0·0 (Grw. mean noon). The moment when L is $18^h\ 40^m$ is the beginning of the Besselian year. For 1900 this is

1900 Jan. 0·313106.

For any other year x it is Jan. 0·0 + k, where − k is the "dies reductus", k being given by:

$$(24)\qquad k = 0^d\cdot313106 + 0^d\cdot24218946\ (x - 1900) - {}^d\cdot000774\ T^2\ \textit{minus}$$
$$\pm\ 28 \qquad \pm\ 42 \qquad \pm\ 71$$

the number of leap years between x and 1900 (not counting x itself).

From (22) the daily motion of the fictitious mean sun is found to be:

$$3548''\cdot2043123 + ''\cdot0000735\ T.$$
$$\pm\ 70 \qquad \pm\ 82$$

The rotation of the earth in a mean solar day consequently is:

$$(25)\qquad \omega = 1299548''\cdot2043123 + ''\cdot0000735\ T.$$
$$\pm\ 70 \qquad \pm\ 82$$

The fundamental unit of astronomical time [1]) is the true sidereal day, which is the interval between two successive meridian passages of an equatorial star without proper motion. The mean solar day is evidently $\omega/1296000''$ true sidereal days. The sidereal day used in astronomical practice is however the interval between two successive meridian passages of the mean equinox. It is equal to $Y/(Y+1)$ mean solar days. The practical sidereal day is $0^s\cdot0083665 + {}^s\cdot00000510\ T$ (mean solar seconds) shorter than the true sidereal day.

As a consequence of the secular retardation of the rotation of the earth the fundamental unit of astronomical time is not constant. It is assumed that the sidereal year, corrected for secular perturbations, is constant, and the time measured in a unit which has a fixed ratio to the sidereal year thus corrected is called "Newtonian time", or "uniform time". The factor by which an interval expressed in astronomical time must be multiplied to express it in Newtonian time is $1 + 2\cdot546 . 10^{-8}\ T$.

[1]) In *B.A.N.* IV, No. 127, p. 38 DE SITTER defined astronomical time as the time given by the earth's rotation, affected by both the secular acceleration and the fluctuations. In the present article the term astronomical time corresponds to what he then called uniformly accelerated time. It is probable that this contradiction in definition was not intentional, but I did not feel justified in carrying out the considerable changes that would have been necessary to make the text conform with the earlier definition of astronomical time. A table for converting astronomical time affected by the fluctuations to "uniformly accelerated time" as used in the present article is given in Table 5 of the article cited (BROUWER).

III. CONSTANTS CONNECTED WITH THE EARTH.

9. *Formulas of geodesy.*

As fundamental constants I will adopt the mean radius R_1 i.e. the radius at the latitude $\varphi = \sin^{-1}\sqrt{1/3}$, the acceleration of gravity g_1 at this radius, the mechanical compression $H = (2C - A - B)/2C$, $A < B < C$ being the moments of inertia, and the small constants $\varkappa$ and λ_1 depending on the inner constitution of the earth.

We will assume $A = B$. The actual earth is thus replaced by an ideal surface of revolution, called the normal surface [2]), which is an ellipsoid with a small depression at the latitude 45°. It is assumed that this normal surface does not differ much from the geoid. The flattening of this ellipsoid is called ε. The equatorial radius is then, correct to the second order of small quantities,

$$(26)\qquad b = R_1\ (1 + \tfrac{1}{3}\,\varepsilon - \tfrac{4}{9}\,\varepsilon^2 + \tfrac{8}{9}\,\varkappa),$$

and the radius at latitude φ is

$$(27)\qquad R = b\ [1 - \varepsilon \sin^2\varphi + (\tfrac{5}{8}\,\varepsilon^2 - \varkappa) \sin^2 2\,\varphi].$$

The acceleration of gravity at the latitude φ

$$(28)\qquad g = g_0\ [1 + \beta \sin^2\varphi + \gamma \sin^2 2\,\varphi].$$

[2]) Cf. *B.A.N.* II, No. 55, p. 101.

The relation between g_1 and g_o is therefore

(29) $$g_1 = g_o \, [1 + \tfrac{1}{3} \beta + \tfrac{8}{9} \gamma].$$

Further, we need the values of

$$J = \tfrac{3}{2} \frac{C - A}{M_1 \, b^2}, \qquad \rho_1 = \frac{\omega^2 \, R_1^3}{f \, M_1},$$

where $f = k^2$ is the gravitation constant and M_1 the mass.

The relations between these different constants are, accurate to the second order of small quantities[1]):

(30) $$\varepsilon = (J + \tfrac{1}{2} \rho_1) \; (1 + J) - \tfrac{1}{6} K,$$

where

$$K = 3 \, \varepsilon \, (J - \tfrac{3}{14} \rho) + \tfrac{24}{7} \varkappa.$$

Therefore

(30′) $$\varepsilon = (J + \tfrac{1}{2} \rho_1) \; (1 + \tfrac{1}{2} J + \tfrac{3}{28} \rho) - \tfrac{4}{7} \varkappa.$$

The theoretical limits of the value of $\varkappa$ are

(31) $$0 \leqq \varkappa \leqq \tfrac{5}{16} \varepsilon\rho - \tfrac{1}{4} \varepsilon^2 = \cdot 00000082;$$

the lower limit corresponds to a homogeneous earth. Since the earth is certainly not homogeneous, it seems better to keep $\varkappa$ in the formulas, instead of putting it equal to zero, as is generally done [2]).

Further we have

(32) $$\rho_1 = \frac{\omega^2 \, R_1}{g_1} \, [1 - \tfrac{2}{3} \rho_1 + \tfrac{14}{9} \varepsilon^2 - \tfrac{10}{9} \varepsilon\rho - \tfrac{16}{9} \varkappa],$$

or complete to the second order

(32′) $$\rho_1 + \tfrac{2}{3} \rho_1{}^2 = \frac{\omega^2 \, R_1}{g_1}.$$

The constants β and γ are given by

(33) $$\begin{aligned} \beta &= \tfrac{5}{2} \rho_1 - \varepsilon - \tfrac{17}{14} \varepsilon\rho + \tfrac{15}{4} \rho^2 + \tfrac{8}{7} \varkappa = \\ &= 2 \, \rho_1 - J - \tfrac{1}{2} J^2 - \tfrac{11}{7} J \rho + \tfrac{173}{56} \rho^2 + \tfrac{12}{7} \varkappa. \end{aligned}$$

(34) $$\begin{aligned} \gamma &= - \tfrac{5}{8} \varepsilon\rho + \tfrac{1}{8} \varepsilon^2 - 3 \varkappa = \\ &= \tfrac{1}{8} J^2 - \tfrac{1}{2} J \rho - \tfrac{9}{32} \rho^2 - 3 \varkappa. \end{aligned}$$

1) Cf. *B.A.N.* II, No. 55, pp. 97–108, 1924. In the terms of the second order the suffix of ρ_1 is omitted, since it is then no longer necessary to distinguish it from $\rho = \omega^2 b^3 / f M_1$.

2) It may be of interest to give some of the usual formulas with $\varkappa$ in them.

The reduction from geographic to geocentric latitude is

$$\varphi' - \varphi = - (\varepsilon + \tfrac{1}{2} \varepsilon^2) \sin 2 \varphi + (\tfrac{1}{2} \varepsilon^2 - 2 \varkappa) \sin 4 \varphi.$$

Radius of the sphere with equal volume is $R_1 \, (1 - \tfrac{2}{3} \varepsilon^2 + \tfrac{16}{45} \varkappa)$, and of the sphere with equal surface $R_1 \, (1 - \tfrac{26}{45} \varepsilon^2 + \tfrac{16}{45} \varkappa)$.

The radii of curvature at the latitude φ are:

in the meridian: $\rho_m = b \, (1 - 2\varepsilon + \varepsilon^2 - 8 \varkappa)$
$\times [1 + (3 \, \varepsilon + 6 \, \varepsilon^2) \sin^2 \varphi - \tfrac{15}{8} (\varepsilon^2 - 8 \varkappa) \sin^2 2 \varphi]$,

normal to the meridian:

$$\rho_n = b \, [1 + (\varepsilon + \varepsilon^2 - 8 \varkappa) \sin^2 \varphi - \tfrac{3}{8} (\varepsilon^2 - 8 \varkappa) \sin^2 2 \varphi].$$

All these formulas are rigorous for the "normal surface". They are independent of the inner constitution of the earth. This only comes in through the relation between J and H.

We have

(35) $$J = q \, H,$$

where

(36) $$q = 1 - \tfrac{1}{3} \rho_1 - \tfrac{2}{5} \, (1 - \tfrac{2}{3} \varepsilon) \frac{\sqrt{1 + \eta_1}}{1 + \lambda_1}.$$

This formula is derived from the theory of CLAIRAUT on the constitution of the earth. It cannot be doubted that, with the exception of the outer crust, this theory is applicable to the actual earth. It has been shown in *B.A.N.* No. 55 that the value of q derived from this theory can be used for the actual earth. It was also shown there that this would remain true even if there were no isostatic compensation in the outer layers. In that case, however, the normal surface would no longer be identical with the geoid.

The value of $1 + \eta_1$ in (36) is given by

(37) $$\begin{gathered} \varepsilon' \, (1 + \eta_1) = 2 \, \rho_1 - J + \tfrac{4}{21} \varepsilon^2 - \tfrac{5}{7} \varepsilon\rho + \tfrac{10}{21} \rho^2, \\ \varepsilon' = \varepsilon - \tfrac{5}{42} \varepsilon^2 + \tfrac{4}{7} \varkappa. \end{gathered}$$

The denominator $1 + \lambda_1$ is a mean value of a certain function, depending on the distribution of density, but never differing much from unity.

10. *Values of fundamental and derived constants.*

I adopt the following fundamental values:

(38) $$\begin{aligned} R_1 &= 6371260 \; (1 + u) \text{ meters} \\ g_1 &= 979{\cdot}770 \; (1 + v) \text{ cm/sec}^2 \\ H &= 0{\cdot}003279423 \; (1 + w) \\ \varkappa &= 0{\cdot}00000050 + 10^{-3} \chi \\ \lambda_1 &= 0{\cdot}00040 + \psi. \end{aligned}$$

The value of R_1 was derived from HEISKANEN's isostatic reductions [3]) of different geodetic surveys. These are

North America	$b = 6378388 \pm 53$		$p = 10$
Europe	397 ± 72		4
India	352 ± 182		1
Africa	358 ± 179		1

The mean with the weights as indicated is 6378386, which, with the provisional value $\varepsilon^{-1} = 296{\cdot}7$, gives the value (38) of R_1.

For g_1 I refer to *B.A.N.* No. 129 (Vol. IV, p. 58, 1927).

The adopted value of H is that corresponding to the value (8) of **P** by the formulas which will be derived below (cf. Section 14). The value of $\varkappa$ is

3) *Veröff. Finn. Geod. Inst.* **6**, 1926.

originally a pure guess inside the theoretical limits (31). The value of λ_1 has been approximated in *B.A.N.* No. 55, p. 100. In that same paper, pp. 107 and 108, I derived the values of $\varkappa$ and λ_1 from two hypotheses regarding the distribution of mass in the interior of the earth in accordance with CLAIRAUT's theory and WIECHERT's hypothesis. The resulting values were

(a) $\cdot 00000063 < \varkappa < \cdot 00000072, \quad \lambda_1 = \cdot 00031$
(b) $\cdot 00000047 \qquad\qquad \cdot 00000052 \qquad\qquad \cdot 00039$

The second hypothesis (gradual increase of density from the isostatic surface downwards to the surface of discontinuity) appears a priori more probable. The adopted values (38) of $\varkappa$ and λ_1 correspond very closely to this hypothesis.

The adopted values of the unknown corrections, u, v, w, χ and ψ are, of course, zero. Their probable errors may be taken to be:

$$(39)\quad \begin{array}{ll} \text{probable error of } u = \pm & 5.10^{-6} \\ \text{,, \quad ,, \quad ,, } v = \pm & 2.10^{-6} \\ \text{,, \quad ,, \quad ,, } w = \pm & 3\cdot 4.10^{-4} \\ \text{,, \quad ,, \quad ,, } \chi = \pm & 10^{-4} \\ \text{,, \quad ,, \quad ,, } \psi = \pm & 10^{-4} \end{array}$$

We then find the values of the other constants by the formulas (26) to (37). The probable errors attached are those corresponding to the probable errors (39) of the fundamental constants.

$$(40)\quad \begin{aligned}
\rho_1 &= \underset{\pm 2}{\cdot 00344993}\ [1 + \cdot 9977\ (u - v)] \\
1 + \eta_1 &= \underset{\pm 74}{1\cdot 56089}\ [1 + 1\cdot 330\ (u - v) - 1\cdot 334\ w - 1\cdot 327\ \psi] \\
q &= \underset{\pm 14}{\cdot 50043}\ [1 - \cdot 6640\ (u - v) + \cdot 6661\ w + 1\cdot 6581\ \psi] \\
J &= \underset{\pm 97}{\cdot 00164112}\ [1 - \cdot 6640\ (u - v) + 1\cdot 6661\ w + 1\cdot 6581\ \psi] \\
\varepsilon &= \underset{\pm 97}{\cdot 00336981} \\
\varepsilon^{-1} &= \underset{\pm 86}{296\cdot 753}\ [1 - \cdot 1874\ (u - v) - \cdot 8138\ w + \cdot 1696\ \chi - \cdot 8098\ \psi]
\end{aligned}$$

The last decimal place is uncertain by one or two units (except in the case of ρ_1) in consequence of the neglect of third order terms in the formulas. It should be noted that in all cases the major part of the probable errors is due to w.

Further

$$(41)\quad \begin{aligned}
b &= 6378387\ \left[1 + u + \cdot 00111\ \frac{\partial\varepsilon}{\varepsilon} + \cdot 00089\ \chi\right] \\
&= \underset{\pm 32}{6378387}\ [1 + 1\cdot 00021\ u - \cdot 00021\ v + \cdot 00090\ w + \cdot 00070\ \chi + \cdot 00090\ \psi] \\
g_0 &= \underset{\pm 20}{978\cdot 0530}\ [1 - \cdot 00267\ u + 1\cdot 00267\ v + \cdot 00092\ w + \cdot 00209\ \chi + \cdot 00092\ \psi] \\
\beta &= \underset{\pm 99}{\cdot 00528612}\ [1 + 1\cdot 5221\ (u - v) - \cdot 5209\ w + \cdot 324\ \chi - \cdot 518\ \psi] \\
\gamma &= -\underset{\pm 30}{\cdot 00000734}\ [1 + 1\cdot 100\ (u - v) + \cdot 491\ w + \cdot 488\ \psi] - \cdot 00299\ \chi \\
R &= \underset{\pm 32}{6378387} - \underset{\pm 6}{21494}\ \sin^2\varphi + \underset{\pm 6}{42}\ \sin^2 2\varphi \\
g &= \underset{\pm 20}{978\cdot 0530} + \underset{\pm 10}{5\cdot 1701}\ \sin^2\varphi - \underset{\pm 3}{\cdot 0072}\ \sin^2 2\varphi \\
\varphi' - \varphi &= -\underset{\pm \cdot 201}{696''\cdot 245}\ \sin 2\varphi + \underset{\pm \cdot 021}{''\cdot 965}\ \sin 4\varphi.
\end{aligned}$$

APPENDIX A

[written by *Dirk Brouwer*, with the aid of notes by *W. de Sitter*].

IV. The solar parallax and other astronomical constants.

11. *Introduction of fundamental constants.*

Let $\pi_\odot$ be the solar parallax, c the velocity of light in vacuo, μ the ratio between the masses of the moon and the earth.

The following values are introduced:

$$\pi_\odot = 8''\cdot8030\ (1 + x)$$
$$(42)\qquad c = 299774\ (1 + y)\ \text{km/sec}$$
$$\mu^{-1} = 81\cdot53\ (1 + z),$$

with

	probable error of x	$\pm\ 2.10^{-4}$
(43)	,, ,, ,, y	$\pm\ 10^{-5}$
	,, ,, ,, z	$\pm\ 5.10^{-4}$

The value for $\pi_\odot$ is generally considered to be the best mean from all determinations now available.

The value for c is that obtained by Michelson, Pease and Pearson [1]) from measurements in a partial vacuum during the years 1929–33.

The value for μ is the one derived by Hinks from the observations of Eros during the opposition 1900–01 [2]), and used in Brown's Tables of the Moon.

In the following sections n, n', ω will be the mean motions of the sun and the moon and the rotation of the earth respectively, expressed in seconds of arc per mean solar day for 1900·0. Thus, from (18), (21), (25):

$$n = 3548''\cdot1928906\ (1 + 2\cdot515.10^{-8}T)$$
$$\pm\ 50 \qquad \pm\ \cdot231$$
$$(44)\quad n' = 47434''\cdot8909701\ (1 + 1\cdot278.10^{-8}T)$$
$$\pm\ 50 \qquad \pm\ \cdot034$$
$$\omega = 1299548''\cdot2043123\ (1 + 0\cdot566.10^{-10}T)$$
$$\pm\ 70 \qquad \pm\ \cdot063$$

The secular terms include the part due to the secular change in the rate of rotation of the earth.

The astronomical unit is defined by

$$1 \text{ astronomical unit} = \frac{b}{\pi_\odot \sin 1''} = \frac{R_1}{\pi_\odot \sin 1''}\left(1 + \tfrac{1}{3}\varepsilon - \tfrac{4}{9}\varepsilon^2 + \tfrac{8}{9}\varkappa\right)$$
$$(45)\qquad = 149453000 \text{ km } [1 - x + 1\cdot0002\,u - \cdot0002\,v + \cdot0009\,w + \cdot0007\,\chi + \cdot0009\,\psi].$$
$$\pm\ 30000$$

Let $a_\circ$ in astronomical units be defined by

$$n^2 a^3_\circ = k^2\ (1 + m),$$

in which m is the mass of the earth-moon system expressed in terms of the sun's mass. Putting

$$a_\circ = 1 + \nu_1 \text{ astronomical units,}$$
$$(46)\qquad k = \cdot0172020985\ ^{3)}$$
$$k'' = 3548''\cdot187607,$$

and taking the values of n, m from (44), (53) respectively, I find

$$\nu_1 = +\ 0\cdot24.10^{-7}.$$
$$\pm\ 1$$

The complete value of the semi-major axis of the earth's orbit is found by adding to $a_\circ$ the constant terms due to the attractions of the planets. The value given by Newcomb [4]), corrected for the changes in the masses, becomes

$$\frac{a}{a_\circ} = 1\cdot000000212.$$
$$\pm\ 4$$

Thus, if

$$(47)\qquad a = 1 + \nu_2 \text{ astronomical units,}$$
$$\nu_2 = +\ 2\cdot36.10^{-7}.$$
$$\pm\ 4$$

12. *The constant of aberration and related constants.*

The expression for **k**, the constant of aberration, is

$$\mathbf{k} = \frac{n\ a \sec \Phi}{86400\ c}$$
$$= \frac{n\ b \sec \Phi}{86400\ c\ \pi_\odot \sin 1''}\ (1 + \nu_2),$$

Φ being the eccentricity angle of the earth's orbit ($e = \sin \Phi$). The light-time is

$$\tau = \frac{b}{c\ \pi_\odot \sin 1''} \text{ seconds of time.}$$

The product $\mathbf{k} c \pi_\odot$ is known with greater accuracy than either **k**, c, or $\pi_\odot$.

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1) *Contr. Mt. Wilson Obs.* No. 552 = *Ap. J.* **82**, p. 26, 1935. A recent determination by *W. C. Anderson* (Abstract in *Phys. Review*, **51**, 596, 1937) gives c = 299764, in good agreement with the adopted value.

2) *M.N.* **70**, p. 73, 1907.

3) If the unit of time is the mean solar day affected by the secular retardation of the earth's rotation, we have

$$k = \cdot0172020985\ (1 + 2\cdot546.10^{-8}\,T).$$
$$\pm\ \cdot230$$

4) *Astr. Pap. A.E.* VI, p. 10, 1898.

$$\mathbf{k}\, c\, \pi_{\odot} = 54036914 \left[1 + \frac{\delta b}{b}\right]$$

$$= \underset{\pm 271}{54036914} \,[1 + 1{\cdot}0002\, u - {\cdot}0002\, v + {\cdot}0009\, w + {\cdot}0007\, \chi + {\cdot}0009\, \psi],$$

(48)
$$\left.\begin{aligned} \mathbf{k} &= \underset{\pm 41}{20''{\cdot}4770} \\ \tau &= \underset{\pm 100}{498^{s}{\cdot}553} \\ &= \underset{\pm 116}{{}^{d}{\cdot}00577029} \end{aligned}\right\} [1 - x - y + 1{\cdot}0002\, u - {\cdot}0002\, v + {\cdot}0009\, w + {\cdot}0007\, \chi + {\cdot}0009\, \psi].$$

The aberration constant has a small secular term

$$- \,''{\cdot}0000143\ T,$$

produced by the secular variation of sec Φ. It is of no practical significance.

The geodesic precession is related to the constant of aberration as follows:

(49)
$$p_g = \tfrac{3}{2}\, \frac{\overline{V^2}}{c^2}\, n$$
$$= \tfrac{3}{2}\, (\mathbf{k} \sin 1''.\ \cos \Phi)^2\, n,$$

$\overline{V^2}$ being the mean value of the square of the earth's orbital velocity. In a tropical century:

(49′)
$$p_g = \underset{\pm 8}{1''{\cdot}9153}\, [1 - 2\,x - 2\,y + 2{\cdot}0004\, u - {\cdot}0004\, v + {\cdot}0018\, w + {\cdot}0014\, \chi + {\cdot}0018\, \psi].$$

13. *The mass of the earth, the lunar parallax, and related constants.*

Let S, E, μE represent the masses of the sun, the earth, and the moon respectively, and let m be the mass of the earth-moon system expressed in terms of the sun's mass, so that

$$E\ (1 + \mu) = m\ S.$$

Since the mass M_1 used in Section 9 does not include the earth's atmosphere, M_1 and E are not identical.

Put

(50)
$$E = M_1\ (1 + \nu_3);\ \nu_3 = +\, 8{\cdot}65.10^{-7}.$$

The relation between m and the solar parallax is found from

(51)
$$g_1 = \frac{f\, M_1}{R_1^{\,2}}\, (1 - \tfrac{2}{3}\, \rho_1 + \tfrac{14}{9}\, \varepsilon^2 - \tfrac{10}{9}\, \varepsilon\rho - \tfrac{16}{9}\, \varkappa),$$

(52)
$$n^2\, a^3_{\circ} = f \{ S + E\ (1 + \mu) \},$$

and (45), (46). The result is

(53)
$$\frac{1 + m}{m}\, \pi^3_{\odot} = \frac{R_1\, n^2}{g_1\ (1 + \mu)}\, \frac{(1 + \nu_1)^3}{(86400)^2 \sin 1''}\, (1 - \nu_3 + \varepsilon - \tfrac{2}{3}\, \rho_1 + \tfrac{5}{9}\, \varepsilon^2 - \tfrac{16}{9}\, \varepsilon\rho + \tfrac{8}{9}\, \varkappa)$$
$$= \underset{\pm 1800}{223705600}\, [1 + {\cdot}0121\ z + {\cdot}9983\ (u - v) + {\cdot}0027\ w + {\cdot}0003\ \chi + {\cdot}0027\ \psi].$$
$$m^{-1} = \underset{\pm 197}{327932}\, [1 - 3x + {\cdot}0121\ z + {\cdot}9983\ (u - v) + {\cdot}0027\ w + {\cdot}0003\ \chi + {\cdot}0027\ \psi].$$

The constant of the *lunar parallax* is defined by

(54)
$$\sin \pi_{☾} = \frac{b}{\mathrm{a}'},$$

in which a′ is the constant of the moon's variation-orbit, first obtained by HILL and used in BROWN's lunar theory.

Let a' be defined by

(55)
$$n'^2\ a'^3 = f\, E\ (1 + \mu),$$

and let

$$\frac{a'}{\mathrm{a}'} = 1 + \nu_4.$$

BROWN's theory gives [1])

$$\nu_4 = +\ {\cdot}000907681.$$

The definition

$$\pi'_{☾} = \frac{\sin \pi_{☾}}{\sin 1''}$$

gives

$$\pi'_{☾} = \frac{R_1}{a' \sin 1''}\, (1 + \tfrac{1}{3}\, \varepsilon - \tfrac{4}{9}\, \varepsilon^2 + \tfrac{8}{9}\, \varkappa)\ (1 + \nu_4).$$

[1]) *Mem. R.A.S.* **53**, p. 89, 1897.

After elimination of a' by (51), (55) the expression becomes:

$$\pi'^3_{☾} = \frac{R_1\, n'^2}{g_1\,(1+\mu)}\,\frac{(1+\nu_4)^3}{(86400)^2 \sin 1''}\,(1 - \nu_3 + \varepsilon - \tfrac{2}{3}\rho_1 + \tfrac{5}{9}\varepsilon^2 - \tfrac{16}{9}\varepsilon\rho + \tfrac{8}{9}\varkappa), \tag{56}$$

$$\pi'_{☾} = 3422''{\cdot}526\,[1 + {\cdot}00404\,z + {\cdot}33277\,(u-v) + {\cdot}00091\,w + {\cdot}00011\,\chi + {\cdot}00091\,\psi].$$
$$\pm 9$$

The *parallactic inequality* is the term

$$-\,P \sin \mathrm{D}$$

in the moon's ecliptic longitude. The value of the coefficient in BROWN's theory is

$$P = 49853''{\cdot}2\,\frac{1-\mu}{1+\mu}\,\frac{\pi_{\odot}}{\pi'_{☾}}$$

$$= 14{\cdot}2132\;\pi_{\odot}\,[1 + {\cdot}0205\,z - {\cdot}3328\,(u-v) - {\cdot}0009\,w - {\cdot}0001\,\chi - {\cdot}0009\,\psi] \tag{57}$$
$$\pm 2$$

$$= 125''{\cdot}119\,[1 + x + {\cdot}0205\,z - {\cdot}3328\,(u-v) - {\cdot}0009\,w - {\cdot}0001\,\chi - {\cdot}0009\,\psi].$$
$$\pm 25$$

Professor BROWN permits me to state that he estimates that the theoretical uncertainty of P, due to the contributions of terms of higher order than those included in his theory, amounts to a probable error $\pm$ ″·003. This corresponds to a probable error $\pm$ 1″·2 in the numerical factor in the first of the above expressions.

For the constant of the *lunar inequality* I introduce

$$L = \frac{\mu}{1+\mu}\,\frac{\pi_{\odot}}{\sin \pi_{☾}}. \tag{58}$$

NEWCOMB, dealing with the lunar inequality in the sun's longitude, introduced a coefficient that will be denoted here by L_s, the coefficient of the term with argument D in the sun's longitude. The ratio L_s/L is, therefore, the coefficient of sin D in

$$\frac{\pi'_{☾}}{\pi_{\odot}}\,\frac{r_{☾}}{r_{\odot}}\cos\beta_{☾}\,\sin\,(\lambda_{☾} - \lambda_{\odot}),$$

developed in terms of the mean anomalies. Its numerical value is

$$\frac{L_s}{L} = 1{\cdot}00450.$$

$$\left.\begin{array}{l} L\;\,= 6''{\cdot}4283 \\ \qquad\;\;\pm 34 \\ L_s = 6''{\cdot}4572 \\ \qquad\;\;\pm 34 \end{array}\right\}[1 + x - {\cdot}9919\,z - {\cdot}3328\,(u-v) - {\cdot}0009\,w - {\cdot}0001\,\chi - {\cdot}0009\,\psi]. \tag{58'}$$

14. *The constants of precession and nutation.*

The expressions for the constants of precession and nutation in terms of the mass of the moon and the dynamical compression of the earth can be written in the form

$$\mathbf{P} = (\mathrm{A} + \mathrm{B}\,\mu')\,H, \qquad \mathbf{N} = \mathrm{C}\,\mu' \cos\Theta\,.\,H, \tag{59}$$

in which

$$\mu' = \mu/(1+\mu).$$

Introduce
$-\alpha$ = the tropical mean motion of the node of the moon's orbit,
Y = the length of the tropical year in mean solar days,
N, N', the functions introduced by HILL [1]), and developed by him with DELAUNAY's lunar theory.

Then

$$\mathrm{A} = \tfrac{3}{2}\,\frac{n^2}{\omega}\,\frac{\sec^3\Phi}{1+m}\cdot\;100\,Y,$$

$$\mathrm{B} = \tfrac{3}{2}\,\frac{n'^2}{\omega}\cdot 100\,YN, \tag{60}$$

$$\mathrm{C} = \tfrac{3}{2}\,\frac{n'^2}{\omega}\cdot\frac{1}{\alpha \sin 1''}\,N'.$$

The expressions for **P**, **N** obtained by introducing (60) into (59) are only in form different from those given on p. 21 of HILL's paper. They give **P** in seconds of arc per tropical century, **N** in seconds of arc.

The numerical values are obtained by using n, n', ω from (44), and

[1]) *Collected Math. Works*, IV, pp. 11–21.

$$\begin{array}{rcll}
\alpha & = & 190''{\cdot}634101 & (1 - 2{\cdot}149.10^{-6}\, T), \\
 & & \pm\ 49 & \quad\pm\ 6 \\
100\, Y & = & 36524^{d}{\cdot}218946 & (1 - 4{\cdot}238.10^{-8}\, T), \\
 & & \pm\ 42 & \quad\pm\ {\cdot}389 \\
\dfrac{\sec^3 \Phi}{1 + m} & = & 1\,{\cdot}000417997 & (1 - 2{\cdot}0917.10^{-6}\, T), \\
 & & \pm\ 9 & \quad\pm\ 60 \\
\cos \Theta & = & {\cdot}91739170 & (1 + 9{\cdot}9020.10^{-5}\, T). \\
 & & \pm\ 12 & \quad\pm\ 174
\end{array}$$

The last three have been obtained from (20), (15), (9′) respectively, and the value for α from the motion of the moon's node per Julian century in Brown's Tables, viz:

$$-\ 6962911''{\cdot}23 + 14''{\cdot}96\ T + ''{\cdot}024\ T^2,$$

to which has been added the correction found by Spencer Jones [1])

$$+\ ''{\cdot}69 \pm 1''{\cdot}78.$$

The resulting values are:

$$\begin{array}{lrcll}
 & A & = & 530977''{\cdot}04 & (1 - 2{\cdot}0839.10^{-6}\, T) \\
 & & & \pm\ {\cdot}05 & \quad\pm\ 85 \\
(61) & B & = & 94858361''{\cdot}2 & (1 - 1{\cdot}69\ .10^{-8}\, T)\ N \\
 & & & \pm\ {\cdot}1 & \quad\pm\ {\cdot}40 \\
 & C & = & 2810083''{\cdot}5 & (1 + 2{\cdot}174\ .10^{-6}\, T)\ N' \\
 & & & \pm\ {\cdot}7 & \quad\pm\ 6
\end{array}$$

The developments of N, N' are taken from Hill's paper; they are, therefore, based upon Delaunay's theory. They were recomputed by substituting the constants of the moon's orbit from Brown's Tables. I find for the contributions of terms of each order separately:

order		order	
0	+ ·99241876	1	+ ·08972741
2	+ 279762	3	+ 25115
4	+ 7045	4	− 1145
5	+ 4949	5	+ 646
6	+ 2420	6	+ 914
7	+ 842	*est.*	+ 429
est. =	+ 266		± 400
	± 50		
N =	+ ·9953716	N' =	+ ·089987
	± 5		± 4

The agreement with Hill is very nearly perfect, the difference in the elliptic parts of N, N' being due to the change in the constants of the lunar orbit. The estimated contribution of the higher order terms in N is relatively certain; an estimate independent of Hill was made in this case. In the case of N' I followed Hill, merely changing the last two figures in order to arrive at zeros for the seventh and eighth digits of N'. Hill did not disclose in detail how he arrived at his estimate, nor how accurate he considered it. In my judgment the probable error of the estimated part equals its full amount.

With these values of N, N',

$$(62)\quad \begin{array}{rcll}
B & = & 94419319'' & (1 - 1{\cdot}69\ .\ 10^{-8}\, T), \\
C & = & 252871'' & (1 + 2{\cdot}174\ .\ 10^{-6}\, T).
\end{array}$$

Finally, introducing the value (38) for H, and (42) for μ^{-1},

$$(63)\quad \begin{array}{rcllcl}
\mathbf{P} & = & 5493''{\cdot}157 & [1 - {\cdot}6747\, z + w] & - & ''{\cdot}003692\ T, \\
 & & \pm\ {\cdot}175 & & & \pm\ 25 \\
\mathbf{N} & = & 9''{\cdot}2181 & [1 - {\cdot}9879\, z + w] & + & ''{\cdot}000933\ T, \\
 & & \pm\ 15 & & & \pm\ 2 \\
p_0 & = & 5093''{\cdot}376 & [1 - {\cdot}6747\, z + w] & + & ''{\cdot}49561\ T, \\
 & & \pm\ {\cdot}161 & & & \pm\ 87 \\
\dfrac{p_0}{\mathbf{N}} & = & 546\ {\cdot}685 & [1 + {\cdot}3131\, z] & - & {\cdot}00156\ T. \\
 & & \pm\ {\cdot}085 & & & \pm\ 1
\end{array}$$

The relatively large secular increase of **N**, nearly $''{\cdot}001$ per century, is almost entirely due to the secular decrease of the obliquity of the ecliptic. Only two percent of its amount has been contributed by the secular term in the motion of the moon's node. The secular term in **P** is almost entirely due to the secular decrease of the eccentricity of the earth's orbit.

The values for the probable errors of **P** and p_0 given in (63) are those of the *observed* quantities; (compare also the footnote on p. 214). It is recalled that the value of H (38) has been so chosen that, with the adopted value for the moon's mass, it agrees exactly with the value of **P** derived from the observed value of p in Section 4, (4), (8) [2]).

The value of **P** derived from observation,

$$\mathbf{P} = 5493''{\cdot}156 \pm ''{\cdot}175\ (1900{\cdot}0),$$

produces the equation

$$-\ {\cdot}6747\ z + w = {\cdot}000000 \pm {\cdot}000032.$$

On account of the smallness of the probable error this represents an almost exact relation which possible corrections z, w must satisfy. The term in square brackets in the expression for **N** may be written as

$$[1 + (w - {\cdot}6747\ z) - {\cdot}3131\ z].$$

1) *Cape Ann.* XIII, part 3, p. 45.

2) There is actually a difference of $0''{\cdot}001$ between the values of **P** in (8) and in (63). In order to arrive at agreement to the last place given it would have been necessary to add an additional significant figure to the value of H.

The errors of the two terms are now independent of each other. The probable error of the term in parentheses is $\pm$ ·00003, that of the last term is $\pm$ ·00016 (as found from the error assigned to z). The total percentage probable error is thus $\pm$ ·00016, corresponding to a probable error of $\pm$ ″·0015 in **N**, as given in (63). The difference between the observed value

$$\mathbf{N} = 9''{\cdot}2075 \pm {''}{\cdot}0020$$

and the computed value given by (63) will be discussed in the next section.

APPENDIX B [by *Dirk Brouwer*]

V. Remarks and Explanatory Notes.

15. *General remarks. Discussion of the observed values for the constant of nutation and the lunar inequality.*

The system of astronomical constants developed by DE SITTER conforms with the requirement that the theoretical relations among the constants are satisfied rigorously. That this should not be understood to mean that the theory is necessarily complete or perfect applies especially to the theoretical relations among the constants of geodesy. JEFFREYS [1]) has remarked that, as DE SITTER's theory does not account for the observed large scale gravity anomalies, the flattening derived may be in error by as much as twenty times its apparent uncertainty. DE SITTER, aware of the limitations of his theory, was nevertheless confident that there were no serious objections to its application to the actual earth.

Minor imperfections of the system are the uncertainties of the numerical factors present in the expressions for the parallactic inequality P, the ratio between the lunar inequality in the Sun's longitude L_s and the constant L, and the value of N' occurring in the expression for the constant of nutation. With the possible exception of N' these uncertainties are of no practical consequence. All three depend on the lunar theory. It is expected that the numerical verification of the solar part of the lunar theory, undertaken by BROWN and ECKERT, will yield a more accurate value for P, and will also make available developments that simplify a new calculation of the other two quantities.

The constant of nutation presents the only serious disagreement in DE SITTER's system between observation and theory. This discrepancy was first discussed by JACKSON [2]), after FOTHERINGHAM had called attention to the fact that the geodesic precession had previously been applied with the wrong sign [3]). JACKSON suggested that the disagreement might be due to imperfections in the theory. DE SITTER gave considerable thought to this problem during the last few years of his life, but with negative results. Among his computations dealing with this subject I found as a summarizing statement that his search for further relativity corrections to the theory of precession and nutation had not revealed any sensible contribution in addition to the geodesic precession. Then followed the remark that this had been derived by making use of the classical equations for the rigid body, i.e. C and A are not changed by the motion itself [4]).

The question how the constants of precession and nutation are affected by the non-rigidity of the earth has been considered by various authors [5]). They conclude that the effect is practically negligible. SCHWEYDAR (l.c. p. 113) finds that the centennial precession is decreased by ″·09 on account of the yielding of the earth.

This examination of the theory appears to leave no other choice but to accept the theoretical expressions for the constants of precession and nutation as they stand. The cause for the discrepancy between the value for the mass of the moon as derived from the lunar inequality and from the constant of nutation must then be sought in possible systematic errors in the observations.

The observed value for the constant of nutation may be taken to be

$$\mathbf{N} = 9''{\cdot}2075 \pm {''}{\cdot}0020\ ^{6)}.$$

This result is the mean of numerous accordant independent determinations. As JACKSON remarks, there is no known reason for suspecting the presence of appreciable systematic errors in these determinations.

For the lunar inequality DE SITTER assumed the observed value

$$L = 6''{\cdot}4283 \pm {''}{\cdot}0029.$$

This is very close to the weighted mean of the separate determinations:

6″·456 $\pm$ ″·012,	(NEWCOMB, Sun)
6″·414 $\pm$ ″·009,	(GILL, Victoria)
6″·431 $\pm$ ″·003,	(HINKS, Eros, 1900–01).

The direct determination from the solar observations has received little attention since NEWCOMB's

[1]) *M.N.* **97**, p. 3, 1936.
[2]) *M.N.* **90**, p. 741, 1930.
[3]) *B.A.N.* IV, No. 129, p. 59, 1927.

[4]) A statement on this question in DE SITTER's "The Astronomical Aspect of the Theory of Relativity", *Univ. of Cal. Publ. in Math.* **2**, No. 8, p. 153, 1933, was probably written before he had completed this investigation.

[5]) KELVIN, *Math. and Phys. Papers* III, pp. 312–350; G. H. DARWIN, *Scientific Papers* II, pp. 36–139; SCHWEYDAR, *A.N.* **203**, pp. 101–116.

I am indebted to Dr. W. D. LAMBERT for these references, and for communicating his views on this subject.

[6]) *B.A.N.* IV, No. 129, p. 59, 1927; *M.N.* **90**, p. 741, 1930.

determination from the right ascensions from early in the nineteenth century till 1864 [1]). The results from the series to which he attaches greatest weight are in satisfactory agreement with each other. Yet, as long as this analysis has not been examined in more detail, it must be considered possible that the results are affected by systematic errors.

The main sources of systematic errors in the determination of the lunar inequality from observations of minor planets were pointed out by HINKS [2]). They are the uncertainty of the ephemeris and of the systematic errors in the positions of the comparison stars. The two sources cannot be separated. Empirical corrections to the ephemeris are derived for the times at which the lunar inequality vanishes, and interpolated for intermediate dates. This interpolation is the more uncertain the smaller the geocentric distance, and consequently, the greater the geocentric arc described by the planet during the period of observation. The uncertainty would appear to be more nearly proportional to the square than to the first power of the reciprocal of the geocentric distance. On that account the possibility of introducing a systematic error into the determination of the moon's mass from observations at a close approach of Eros is not negligible, although I found it difficult to judge from HINKS's presentation to what extent his result may be affected.

In GILL's solution from Victoria the circumstances were decidedly more favorable, the ephemeris being more accurate on account of the greater geocentric distance, and the planet remaining within a small range of both right ascension and declination during the entire period of observation. The systematic error is probably small compared with the accidental error. I, therefore, conclude that the only determination of the lunar inequality for which the probable error may be considered to be a true measure of the uncertainty is GILL's:

$$L = 6''\cdot414 \pm ''\cdot009.$$

With a better determined orbit and improved star positions the effects of both sources of systematic errors in the Eros campaign 1930–31 were probably smaller than in the campaign thirty years earlier, but it is desirable to redetermine the lunar inequality from observations of minor planets comparable with Victoria, whatever result may be obtained from the Eros observations during the opposition 1930–31.

The following three solutions are given:

Solution I is that used by DE SITTER. The mass of the moon is derived from $L = 6''\cdot4283 \pm ''\cdot0029$. This gives the equation

$$x - \cdot9919\, z - \cdot0009\, w = \cdot00000 \pm \cdot00046,$$

or, by elimination of x and w by means of the adopted values for the solar parallax and the constant of precession

(64) $$z = \cdot00000 \pm \cdot00050.$$

The computed value for **N** differs from the observed value by five times the probable error.

Solution II is made by deriving the mass of the moon from the observed value for the nutation constant, $\mathbf{N} = 9''\cdot2075 \pm ''\cdot0020$. With $p_0 = 5039''\cdot376 \pm ''\cdot161$ the observed ratio becomes:

$$p_0/\mathbf{N} = 547\cdot312 \pm \cdot120.$$

Therefore, from (63),

$$+\cdot3131\, z = +\cdot00115 \pm \cdot00020,$$

(65) $$z = +\cdot00367 \pm \cdot00070.$$

The computed value for L differs from DE SITTER's adopted value by eight times the assigned probable error.

Solution III is made by combining

(66) $$z = +\cdot00224 \pm \cdot00142,$$

obtained from GILL's value for the lunar inequality, with equation (65) obtained from the observed value for the constant of nutation.

This gives

(67) $$z = +\cdot00337 \pm \cdot00063.$$

Solutions	I	II	III
w	0	$+\cdot00248$	$+\cdot00227$
z	0	$+\cdot00367$	$+\cdot00337$
L	$6''\cdot4283$	$6''\cdot4049$	$6''\cdot4068$
N	$9''\cdot2181$	$9''\cdot2075$	$9''\cdot2083$
μ^{-1}	81 ·530	81 ·829	81 ·805
ε^{-1}	296 ·753	296 ·155	296 ·205

All three solutions have been made without introducing as additional unknowns corrections to the solar parallax and the constant of precession, i.e. with $x = 0$, $-\cdot6746\, z + w = 0$. Such a complication of the solutions was found to be unnecessary; a solution in which x was included as an additional unknown gave a correction $+''\cdot0017$ to the solar parallax and produced changes of only $''\cdot0002$ in L and **N**. No plausible change in the constant of precession would materially alter the solutions. OORT in his latest discussion of galactic rotation finds to the value for p_0 here adopted a correction $+''\cdot51$ [3]). This may be the best correction available at the present time. It would increase the computed value for the constant of nutation by $''\cdot0009$.

In Solution III, which I consider the most satisfactory, the discrepancy between the lunar inequality and the constant of nutation has been

[1]) *Astronomical Constants*, p. 141. [2]) *M.N.* **70**, p. 63, 1909.

[3]) *B.A.N.* VIII, No. 298, p. 149, 1937.

removed completely. Pending further determinations of the lunar inequality I prefer a slight modification of this solution, obtained by adopting for **N** the value in use at the present time. Leaving the other fundamental constants of DE SITTER's system unchanged, the system is defined by:

$u = v = \chi = \psi = x = y = 0,$

$w = +\,{\cdot}00190 \pm {\cdot}00050, \quad H = {\cdot}003285665,$

$z = +\,{\cdot}00282 \pm {\cdot}00070, \quad \mu^{-1} = 81{\cdot}760 \pm {\cdot}060.$

This system is consistent with the following adopted values for L, **N**:

$$L = 6''{\cdot}4103 \pm 90, \qquad \mathbf{N} = 9''{\cdot}2100 \pm 20$$

$$\text{or: } L_s = 6''{\cdot}4392 \pm 90, \qquad p_0/\mathbf{N} = 547{\cdot}168 \pm 120$$

The values for $\pi_\odot$, **P**, p_0, etc. are the same as in DE SITTER's system. Some of the more important derived constants become:

$$q = {\cdot}501062 \pm 186, \qquad \beta = {\cdot}00528088 \pm 41,$$

$$J = {\cdot}00164632 \pm 140, \qquad \gamma = -{\cdot}00000735 \pm 30,$$

$$\varepsilon^{-1} = 296{\cdot}294 \pm {\cdot}123, \qquad m^{-1} = 327945 \pm 197,$$

$$b = 6378398 \pm 32, \qquad \pi'_{\text{☾}} = 3422''{\cdot}571 \pm 12,$$

$$g_0 = 978{\cdot}0547 \pm 20, \qquad P = 125''{\cdot}126 \pm 25$$

16. *Explanatory notes.*

At the time of Professor DE SITTER's death, on November 20, 1934, the first three chapters of his article had been written. Of the fourth chapter only a brief list of contents of the intended sections existed. Most of the calculations for this fourth chapter had, however, been completed. The results had been gathered on the final page of the note-book in which most of the computations in connection with this article had been made. In writing the fourth chapter I have made extensive use of the contents of this note-book.

The first three chapters as printed are identical with the manuscript with the exception of such corrections as were found necessary. I have completely checked all formulas and practically all numerical results. The corrections applied are explained below. I make no mention of a number of minor alterations either in wording or in numerical values; these are all matters of detail that do not affect any of the essential contents of the paper.

Notes to Chapter I.

Before I undertook the preparation for publication of the manuscript it had been examined at the Leiden Observatory. This work had revealed a slight error in the quantities $\varkappa \sin L$, $\varkappa \cos L$, in Section 3. The correction to NEWCOMB's mass of Mars had been used with the wrong sign. The same error was present in the values for de/dT', $ed\varpi/dT'$ in Chapter II, Section 6.

It had also been noticed that some inconsistency existed between the expressions of the precessional quantities (section 5) for 1850 and 1900. In the course of recomputation I found that this was due to the fact that the values given in the manuscript as p, p_1, etc. actually corresponded to ANDOYER's $d\psi/dt$, $d\omega/dt - d\varphi/dt$, etc., whereas they should correspond to the coefficients of θ in ANDOYER's (ψ), $(\omega) - (\varphi)$, etc. The difference between the two expressions is pronounced in the coefficients of T^2, and also appreciable in the coefficient of T in λ. The expressions for the precessional quantities in powers of the time as printed are the result of a complete recomputation in which the corrected values for $\varkappa \sin L$, $\varkappa \cos L$ were used. The change in the latter produced a change in **P** of only $+\,''{\cdot}010$. The probable errors attached to the coefficients were recomputed and corrected where necessary. The last place of the coefficients in the expressions for $\pi \sin \Pi$, $\pi \cos \Pi$ is, moreover, uncertain on account of the rounding-off present in NEWCOMB's published values for $\varkappa \sin L$, $\varkappa \cos L$.

Notes to Chapter II.

This chapter was in almost perfect condition. Some slight alterations were necessary on account of the corrections to the expressions of the precessional values. For the difference between the practical sidereal day and the true sidereal day the manuscript gave $^s{\cdot}0084124$, the correct value being $^s{\cdot}0083665$. The cause of this discrepancy has not been ascertained.

The value for $d\varpi/dT'$ given by (16) represents only the portion of the sidereal motion of the sun's perigee that is due to the Newtonian attraction of the planets. The tropical motion per Julian century for 1900 is obtained as follows:

planets, Newtonian	$+\ 1152''{\cdot}33$	$+\ 1''{\cdot}07\ T$	$+\ ''{\cdot}073\ T^2$
moon	$+\quad 7\ {\cdot}68$		
relativity	$+\quad 3\ {\cdot}83$		
sidereal motion	$+\ 1163''{\cdot}84$	$+\ 1''{\cdot}07\ T$	$+\ ''{\cdot}073\ T^2$
general precession	$+\ 5026\ {\cdot}11$	$+\ 2\ {\cdot}23$	$+\ {\cdot}000$
tropical motion	$+\ 6189''{\cdot}95$	$+\ 3''{\cdot}30\ T$	$+\ ''{\cdot}073\ T^2$

The resulting tropical motion of the sun's perigee is very little different from that used in NEWCOMB's tables of the sun, the constant term of which is 6189″·03. This agreement is mainly due to the fact that the replacement of NEWCOMB's empirical term (+ 10″·45) by the relativity term (+ 3″·83) is very nearly cancelled by the increased value of the principal part due to changes in the planetary masses.

Notes to Chapter III.

Not a single change was necessary in section 9 with the exception of a few obvious corrections in two formulae. Owing to the small change in **P** the value for H was diminished by two units in the last place given.

The differential factors in (40), (41) required considerable changes. It appeared that DE SITTER had, up to a very late stage in the preparation of the article, used ε^{-1} as a fundamental constant, and had developed the differential factors in this form. In transforming the formulae to H as a fundamental constant errors had crept in.

In the computation of the probable errors that of w had been given the value $\pm$ 3.10^{-5}, though, when ε^{-1} was used, the probable error was stated as $\pm \frac{1}{3}.10^{-3}$. The latter is of the correct order of magnitude, and has been adopted. The manuscript stated that the major part of the probable errors in the derived constants is due to ψ. This would be correct with the probable error of w reduced by a factor 1/10.

Notes to the Appendix, Chapter IV.

The values for $\pi_\odot$, c, μ^{-1} and their probable errors are those adopted by DE SITTER. Most of his earlier computations had been made with c = 299800 or 299796; the change to the finally adopted value was made at a late stage of the work. Although in my opinion the probable errors assigned to c, μ^{-1} are too small, I have continued with DE SITTER's values.

The formula (49) for p_g was added by me; the expression (6) in section 4 of the manuscript lacked the factor c^{-2}, and it is not in the desired form for numerical evaluation.

The values for ν_1, ν_2, ν_3, ν_4 are those used by DE SITTER. An independent recomputation gave identical values for ν_1, ν_2, ν_4.

The expressions for the mass of the earth and for the lunar parallax were found among DE SITTER's computations. In the former his computations had been made with the value $-2{\cdot}12.10^{-7}$ instead of $\nu_1 = +0{\cdot}24.10^{-7}$. This would correspond to the definition $\sin \pi_\odot = b/a$ instead of that given in (45), 1 astr. unit = $b/\sin \pi_\odot$. The distinction between the two definitions is of no numerical importance. Both have been used in the past, and have frequently been treated as if they were identical. It appears to me that the definition used in this article is more expedient and, possibly, more logical.

Some difficulty was encountered with the lunar inequality. It was apparently DE SITTER's intention to follow NEWCOMB in defining L_s as the constant of the lunar inequality. Among his computations no explicit statement as to the ratio L_s/L was found. From his numerical results I concluded that the ratio 1·00450 had been used. The source of this value is unknown to me. NEWCOMB's value (*Astr. Const.*, p. 189) was 1·00460, whereas GILL (*Cape Ann.* VI, part 6, p. 21) gave 1·00445. BAUSCHINGER (*Enc. Math. Wiss.* VI, 2A, p. 857) ignored the distinction between L and L_s. The disagreement is of little practical significance; it appears desirable, however, to adopt L, defined by (58), as the constant of the lunar inequality.

The arrangement of Section 14 on the constants of precession and nutation has been made to conform as closely as possible with what appeared to be DE SITTER's plan. The computations in his notebook were not complete and not in a definitive form.

DE SITTER had taken the values of N, N' from HILL, without corrections to the constants, which are numerically unimportant. Since I thought it desirable to repeat the numerical part of HILL's work I introduced the revised lunar constants. The contributions due to terms of each order separately have been given in order to call attention to the relatively large uncertainty in N' due to the slow convergence of DELAUNAY's developments.

After some hesitation I decided to add the final Section 15 for which I am alone responsible. DE SITTER would have added a discussion on the contradiction between the observed values for the lunar inequality and the constant of nutation, but there was no indication that he had arrived at a definite conclusion or even at a plan for presenting the subject. In writing this section I had, therefore, to rely entirely upon my own views on the subject.

The preparation of this article for publication was undertaken at the request of Prof. J. H. OORT. I am indebted to him for his active interest in the work, for reading the completed manuscript, and for his helpful criticism.

Yale University Observatory

New Haven, Conn.

1937, December

TABLE OF ASTRONOMICAL CONSTANTS [1])

ADOPTED FUNDAMENTAL CONSTANTS

				probable errors	generally adopted value
mean radius of the earth,	R_1	=	6371260 $(1 + u)$ meters	of u, ± 5.10^{-6}	6371269 (Intern. Ell. [2]))
acc. of gravity in mean latitude,	g_1	=	979·770 $(1 + v)$ cm/sec^2	,, v, ± 2.10^{-6}	979·768 (Stockholm, 1930)
dynamical flattening, $(C—A)/C$,	H	=	·003279423 $(1 + w)$	,, w, ± $3·4.10^{-4}$	—
constants depending on the	$\varkappa$	=	·00000050 + $10^{-3}\chi$	,, χ, ± 10^{-4}	$\varkappa = 0$
inner constitution of the earth,	λ_1	=	·00040 + ψ	,, ψ, ± 10^{-4}	—
solar parallax,	$\pi_\odot$	=	8″·8030 $(1 + x)$	,, x, ± 2.10^{-4}	8″·80 (Paris, 1896)
velocity of light,	c	=	299774 $(1 + y)$ km/sec	,, y, ± 10^{-5}	299860 (Newcomb)
reciprocal of moon's mass,	μ^{-1}	=	81·53 $(1 + z)$	,, z, ± 5.10^{-4}	81·53 (Hinks)

MEAN MOTIONS PER MEAN SOLAR DAY

(T = tropical centuries counted from 1900·0)

				generally adopted value
earth's mean motion,	n	=	3548″·1928906 ± 50 + ″·0000892 ± 82 T	·192832 (Newcomb)
moon's mean motion,	n'	=	47434″·8909701 ± 50 + ″·000606 ± 16 T + ″·00000056 T^2	·890654 (Brown)
rotation of the earth,	ω	=	1299548″·2043123 ± 70 + ″·0000735 ± 82 T	
tropical year,	Y	=	365^d·24218946 ± 42 — ^{d}·00001548 ± 142 T	·24219879 (Newcomb)
sidereal year,			365^d·25635442 ± 50 — ^{d}·00000918 ± 84 T	·25636042 (Newcomb)
general precession/tropical century,	p	=	5026″·00 ·10 + 2″·2337 T + ″·00014 T^2	5025″·64 (Newcomb)
planetary precession in longitude/tropical century,	$\lambda \cos \Theta$	=	11″·461 ± ·125 — 1″·7381 T — ″·00021 T^2	11″·44 (Newcomb)
obliquity of the ecliptic,	Θ	=	23°27′8″·29 ± ·07 — 47″·080 T — ″·0059 T^2 + ″·00186 T^3	23°27′8″·26 (Newcomb)
eccentricity of earth's orbit,	e''	=	3455″·169 ± ·035 — 8″·583 T — ″·0262 T^2 — ″·00011 T^3	3455″·150 (Newcomb)

DERIVED CONSTANTS

					generally adopted value
$\omega^2 R_1^3/fM_1$ =	ρ_1	=	·00344993 ± 2	$[1 + ·9977\,(u—v)]$	
$\frac{3}{2}\,C/M_1 b^2$ =	q	=	·50043 ± 14	$[1 — ·6640\,(u—v) + ·6661\,w + 1·6581\,\psi]$	
$\frac{3}{2}\,(C—A)/M_1 b^2$ =	J	=	·00164112 ± 97	$[1 — ·6640\,(u—v) + 1·6661\,w + 1·6581\,\psi]$	
reciprocal of oblateness,	ε^{-1}	=	296·753 ± 86	$[1 — ·1874\,(u—v) — ·8138\,w + ·1696\,\chi — ·8098\,\psi]$	297·0 (Intern. Ell.)
equatorial radius, (meters),	b	=	6378387 ± 32	$[1 + 1·00021\,u — ·00021\,v + ·00090\,w + ·00070\,\chi + ·00090\,\psi]$	6378388 (Intern. Ell.)

acceleration of gravity, (cm/sec²),	g	$= g_0\ [1 + \beta \sin^2 \varphi + \gamma \sin^2 2\varphi]$,	
	g_0	$= 978.0530\,[1 - .00267\,u + 1.00267\,v + .00092\,w + .00209\,\chi + .00092\,\psi]$ ± 20	978·049 (Stockholm, 1930)
	β	$= .00528612\ [1 + 1.5221\ (u-v) - .5209\,w + .324\,\chi - .518\,\psi]$ ± 99	·0052884 (Stockholm, 1930)
	γ	$= -.00000734\ [1 + 1.100\ (u-v) + .491\,w + .488\,\psi] - .00299\,\chi$ ± 30	— ·0000059 (Stockholm, 1930)
radius of the earth in latitude φ, (meters),	R	$= 6378387 - 21494 \sin^2\varphi + 42 \sin^2 2\varphi$ $\pm 32 \quad \pm 6 \quad \pm 6$	
acceleration of gravity in latitude φ, (cm/sec²),	g	$= 978.0530 + 5.1701 \sin^2\varphi - .0072 \sin^2 2\varphi$ $\pm 20 \quad \pm 10 \quad \pm 3$	
reduction from geographic to geocentric latitude,	$\varphi' - \varphi$	$= -696''.245 \sin 2\varphi + ''.965 \sin 4\varphi$ $\pm .201 \quad \pm .021$	
astronomical unit, (km),		$149453000\ [1 - x + 1.0002\,u - .0002\,v + .0009\,w + .0007\,\chi + .0009\,\psi]$ ± 30000	
constant of aberration,	$\mathbf{k}$	$= 20''.4770$ ± 41	20″·47 (Paris, 1896)
light-time,	τ	$= 498^s.553$ $\pm .100$ $^d.00577029$ ± 116 (k and τ:) $[1 - x - y + 1.0002\,u - .0002\,v + .0009\,w + .0007\,\chi + .0009\,\psi]$	498ˢ·58
mass-ratio, m = (earth + moon)/sun,	m^{-1}	$= 327932\ [1 - 3x + .0121\,z + .9983\,(u-v) + .0027\,w + .0003\,\chi + .0027\,\psi]$ ± 197	329390 (Newcomb)
constant of sine of lunar parallax, sin $\pi_☾$/sin 1″,	$\pi'_☾$	$= 3422''.526\ [1 + .00404\,z + .33277\,(u-v) + .00091\,w + .00011\,\chi + .00091\,\psi]$ $\pm .009$	3422″·54 (Brown)
parallactic inequality in moon's longitude,	P	$= 125''.119\ [1 + x + .0205\,z - .3328\,(u-v) - .0009\,w - .0001\,\chi - .0009\,\psi]$ $\pm .025$	125″·154 (Brown)
constant of lunar inequality,	L	$= 6''.4283$ ± 34	6″·425 (Newcomb)
lunar inequality in sun's longitude,	L_s	$= 6''.4572$ ± 34 (L and L_s:) $[1 + x - .9919\,z - .3328\,(u-v) - .0009\,w - .0001\,\chi - .0009\,\psi]$	6″·454 (Newcomb)
constant of precession,	$\mathbf{P}$	$= 5493''.157\ [1 - .6747\,z + w] - ''.003692\,T$ $\pm .175$	5490″·66 (Newcomb)
constant of nutation,	$\mathbf{N}$	$= 9''.2181\ [1 - .9879\,z + w] + ''.000933\,T$ ± 15	9″·210 (Paris, 1896)
lunisolar precession,	p_0	$= 5039''.376\ [1 - .6747\,z + w] + ''.49561\,T$ $\pm .161$	5037″·08 (Newcomb)

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1) The probable errors in the table are those derived from the assumed probable errors of u, v, w, etc. except in the case of **P**, **N** and p_0. See the footnote on p. 214 and the remark at the end of Section 14.

2) The dimensions of the earth in this column are those of the International Ellipsoid of Reference, adopted by the Section of Geodesy of the International Union of Geodesy and Geophysics, meeting at Madrid in 1924. This is substantially Hayford's ellipsoid. The gravity formula was adopted at Stockholm in 1930.